Technical Notes

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In-Plane Functions of Circular Sector Plates

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CIRCULAR sector plates form the decks of circularly curved bridges.[1] For small deformations, in- and out-of-plane behaviors are uncoupled. Knowledge of a variation of cross-sectional elements and displacements throughout the plate area is necessary for design purposes. Elsewhere[2] equations were derived in terms of initial parameters for the case of bending. This Note deals with the plate's in-plane behavior.

Initial parameter solutions[3] appear in literature under other titles, known as Macaulay's method[4] and transfer matrices.[5,6] The advantage of these approaches lies in their generality of tackling arbitrary boundary and loading situations, and even certain types of cross-sectional variations[7] of two-point boundary value problems. Their ability to define a behavior across load discontinuities and abrupt cross-sectional variations by a single equation is especially noteworthy.

Figure 1 shows a circular sector slab supported along its radial edges by diaphragms. The diaphragm is assumed to be infinitely stiff within its plane, but can offer little or no resistance normal to the plane. The plate is acted upon by concentrated radial force Pr_L at point $L(r_L, \phi_L)$, tangential forces $P\phi_L$ along the arc of radius r_L assumed symmetrical with respect to the sector's centroidal radial line, and by normal and shear forces Nr_i, Υ_i, and Nr_o, Υ_o along the inner and outer curved boundaries, respectively. An element with cross-sectional elements—normal and shear forces Nr, $N\phi$, and Υ—and radial and tangential force-load intensities pr, $p\phi$ is also sketched in the same figure. Deflections U, V in the radial and tangential directions are considered positive, as shown in Fig. 1. The angle α subtended at the center of curvature of the sector is assumed to be less than π rad.

Denoting $\alpha_m = m\pi/\alpha$, where m is a positive integer, the deflections U, V, expressed in the trigonometric series form

$$U=\sum_m U_m \sin\alpha_m\phi \qquad V=\sum_m V_m \cos\alpha_m\phi \tag{1a}$$

satisfy the radial edge boundaries' diaphragm support conditions.

Corresponding series of the involved actions are

$$f=\sum_m f_m \sin\alpha_m\phi \qquad f=Nr, N\phi, pr, Pr \tag{1b}$$

$$g=\sum_m g_m \alpha_m\phi \qquad g=\Upsilon, p\phi, P\phi \tag{1c}$$

where the coefficients f_m, g_m, like U_m, V_m in Eq. (1a), are functions of radial coordinate r only.

The coefficients of the concentrated load series in Eqs. (1b) and (1c) are

$$Pr_{Lm} = (2Pr_L/\alpha r_L)\sin\alpha_m\phi_L$$

$$P\phi_{Lm} = \sum_{\phi_L} (2P\phi_L/\alpha r_L)\cos\alpha_m\phi_L \tag{1d}$$

The well known stress-strain and strain-displacement relations and the element equilibrium equations[8] in terms of the mth coefficients of the actions' and deflections' series are rewritten as

$$\begin{bmatrix} r\,Nr/F \\ r\,\Upsilon/F \\ r^2 pr/F \\ r^2 p\phi/F \\ rN\phi/F \end{bmatrix}_{\rho m} = \begin{bmatrix} \nu/\nu_- & -\nu\alpha_m/\nu_- & 0 \\ \alpha_m/2 & -1/2 & 0 \\ 1/\nu_- & -\alpha_m/\nu_- & \alpha_m \\ -\alpha_m/\nu_- & \alpha_m^2/\nu_- & -1 \\ 1/\nu_- & -\alpha_m/\nu_- & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ r\,\Upsilon/F \end{bmatrix}_{\rho m}$$

$$+ \begin{bmatrix} \rho/\nu_- & 0 & 0 \\ \rho/2 & 0 & 0 \\ \nu\rho/\nu_- & -\rho & 0 \\ -\nu\alpha_m\rho/\nu_- & 0 & -\rho \\ \nu\rho/\nu_- & 0 & 0 \end{bmatrix} \begin{bmatrix} (U)^{\cdot} \\ (r\,Nr/F)^{\cdot} \\ (r\,\Upsilon/F)^{\cdot} \end{bmatrix}_{\rho m} \tag{2}$$

where $F=Et/(1+\nu)$; E,t,ν = modulus of elasticity, thickness, and Poisson's ratio of the plate; $\rho = r/r_I$ is the nondimensional radial coordinate; and r_I, arbitrarily chosen, is the radius associated with the initial parameters U_{Im}, V_{Im}, Nr_{Im}, and Υ_{Im}. Inner and outer boundary radii r_i, r_o are special cases; the subscript ρm indicates mth coefficient of the quantity considered at the radial location ρ; $(\;)^{\cdot}$ denotes differentiation with respect to ρ of the therein enclosed quantity; and $\nu_- = 1-\nu$.

The concentrated load coefficients Pr_{Lm} and $P\phi_{Lm}$ are expressed as distributed force intensities in terms of the singularity functions[4]:

$$pr_{\rho m} = s(Pr_{Lm}/r_I)\,\dot{u}_s\langle\rho-\rho_L\rangle$$

$$p\phi_{\rho m} = s(P\phi_{Lm}/r_I)\,\dot{u}_s\langle\rho-\rho_L\rangle \tag{3}$$

where s is -1 for $r>r_L>r_I$, $+1$ when $r<r_L<r_I$, and 0 at all the points on the r_I side of r_L; $\rho_L = r/r_L$; $\dot{u}_s\langle\rho-\rho_L\rangle$, the first derivative with respect to ρ of the unit step function u_s, is the unit impulse function at ρ_L; and Macaulay's parentheses $\langle\;\rangle$ have a special meaning: $\langle\;\rangle^m = 0$ for points on the r_I side of r_L, and $=(\;)^m$ at other locations.

Changing the independent variable ρ and η according to $\eta = Ln\ \rho$ facilitates solution of the involved differential equations by Laplace transformation. Thus, the first four rows of Eq. (2) may be solved in terms of Eqs. (3) for $U_{\rho m}$, $V_{\rho m}$, $(r\,Nr_{\rho m}/F)$, and $(r\,\Upsilon_{\rho m}/F)$. Switching back to ρ, and substituting the resulting solutions into the fifth row of Eq. (2), then yields the influence values for the deflections and

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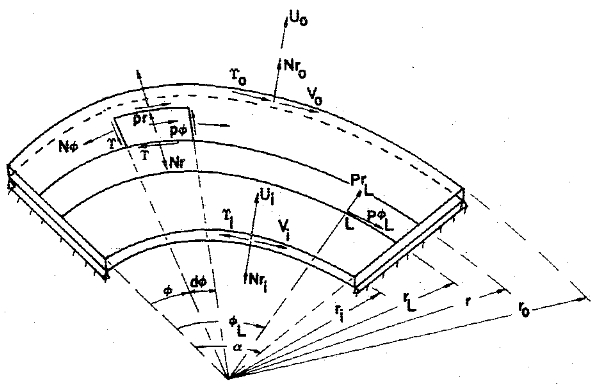

Fig. 1 Circular sector plate and its element.

normal and shear forces in terms of the initial parameters $U_{lm}, V_{lm}, Nr_{lm}, \Upsilon_{lm}$, and loads $Pr_{Lm}, P\phi_{Lm}$:

$$\begin{bmatrix} U \\ V \\ r\,Nr/F \\ r\,\Upsilon/F \\ r\,N\phi/F \end{bmatrix}_{\rho m} = \underbrace{\begin{bmatrix} A_- & B_- & C_- & D_+ \\ F_- & B_+ & D_- & C_+ \\ E_+ & G_+ & A_+ & F_+ \\ G_- & E_- & H_- & H_+ \\ J_+ & J_- & I_+ & I_- \end{bmatrix}}_{T_{\rho l}} \underbrace{\begin{bmatrix} \bar{C}_- & \bar{D}_+ \\ \bar{D}_- & \bar{C}_+ \\ \bar{A}_+ & \bar{F}_+ \\ \bar{H}_- & \bar{H}_+ \\ \bar{I}_+ & \bar{I}_- \end{bmatrix}}_{T_{\rho L}} \begin{bmatrix} U_{lm} \\ V_{lm} \\ r_l\,Nr_{lm}/F \\ r_l\Upsilon_{lm}/F \\ \hline s r_L Pr_{Lm}/F \\ s r_L P\phi_{Lm}/F \end{bmatrix} \tag{4}$$

where matrices $T_{\rho l}$, $T_{\rho L}$, respectively, transport the mth coefficients of the initial parameters at the arc-line of radius r_l, and loads at the arc-line of radius r_L into deflections and normal and shear forces along the arc-line of radius r. Denoting $b_\pm = (\rho^{\alpha_m} \pm \rho^{-\alpha_m})/8$, $c_\pm = \rho \pm 1/\rho$, $\alpha_{m2} = 1/(\alpha_m^2 - 1)$, $\nu_\pm = 1 \pm \nu$, $\nu_2 = 3 - \nu$, $d_\pm = \alpha_m \nu_+ c_\pm$, $d_m = \alpha_m d_-$, $x_- = \nu_-$, $x_+ = 2$, $y_- = 0$, $y_+ = \nu_+$, $z_- = \rho$, $z_+ = c_+$, $e_\pm = d_m \mp 2x_\pm c_-$,

$$A_\pm = 2(\nu_\pm \rho + \nu_\mp/\rho) b_+ - d_- b_-$$

$$B_\pm = d_- b_\mp \pm 2x_\pm \rho b_\pm$$

$$C_\pm = \alpha_{m2}(\alpha_m \nu_2 c_+ b_- \pm e_\pm b_+)$$

$$D_\pm = \alpha_{m2}[(2\nu_2 \rho^{\pm l} \pm e_+) b_- - \alpha_m \nu_2 c_- b_+]$$

$$E_\pm = (2y_\pm c_- \mp d_m) b_+ + d_+ b_-$$

$$F_\pm = (\pm 2\nu_- \rho^{\pm l} - 4c_-) b_- \pm d_- b_+$$

$$G_\pm = -d_- b_+ \pm (d_m \mp 2\nu_+ \rho^{\pm l}) b_-$$

$$H_\pm = 2x_\pm b_\pm/\rho \pm d_- b_\mp$$

$$I_\pm = \pm 2(\nu_+ \rho \mp x_\mp/\rho) b_\pm \pm d_- b_\mp$$

$$J_\pm = \pm(d_m + 2\nu_+ z_\pm) b_\pm \pm (d_\mp + 2\alpha_m \nu_+ \rho) b_\mp \tag{5}$$

where when the double signs $\pm$, $\mp$ occur, the upper sign and symbols with the upper sign belong in one equation, and those corresponding to the lower sign in another equation.

The elements in the Pr and $P\phi$ columns of Eq. (4) are, respectively, obtained from those of Nr and Υ columns by simply replacing ρ with $\langle\bar{\rho}\rangle$, where it occurs explicitly and also in the notations $b_\pm$, $c_\pm$, $z_\pm$ used in Eqs. (5) (note that $\bar{\rho} = r/r_L$).

By choosing the initial parameters at the outer edge ($r_l = r_o$), the values of both ρ and $\bar{\rho}$ are limited to 1, thus avoiding large numbers in the numerical computation process.

Initial parameters approach enables complete definition of a behavior across the load discontinuities by a single equation. Knowing the parameters, the given equations enable evaluation of the distribution of in-plane deflections and normal and shear forces across the plate. The expressions will also readily facilitate solution for arbitrary conditions along the curved boundaries.

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Hydraulic Equivalent of Generalized One-Dimensional Flow

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Introduction

THE similarity between the flow of a compressible gas and the flow of shallow water with a free surface is referred to as the hydraulic analogy. Since its development by Preiswerk,[1] the analog has been used successfully to study qualitatively both internal[2,3] and external[4-6] gas flowfields by flowing water in a rectangular channel. Bryant[7] has shown that this channel must have a constant rectangular shape if the analog is to be valid for a two-dimensional gas flow. Although he demonstrated this fact for an assumed isentropic gas, the same channel requirements exist if the flow is nonisentropic. Furthermore, his assumption of constant entropy leads to the conclusion that the water actually represents a fictitious gas with a specific heat ratio γ equal to 2.

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